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Static traffic assignment with side constraints in a dense orthotropic network

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Abstract

This paper proposes an investigation of bidimensional modeling for traffic flows in a large and dense network of an urban area. The network is viewed as a continuum of orthotropic roads. This (and the side-constraints) constitutes the main research contributions of the paper. Each point of the area is characterized by a travel cost and a side constraint that depend on the privileged directions. A side constraint is a means to obtain a more precise description of traffic flow as far as it does not allow it to exceed the capacity of the roads (Larsson & Patriksson, 1995; Taguchi & Iri, 1982). Each destination is characterized both by a monotonic strictly decreasing function to reflect the customers' elastic demand with respect to the total cost (Yang & Wong, 2000) and a satisfaction function to reflect the customers' gain as entering the desired destination. The study deals with a multi-commodity traffic assignment. We suppose that there are a few destinations for the commuters in the area. All the commuters are uniformly distributed in the area and try to reach their destination. A commodity is defined as the traffic flows from the area to one of the destinations (Yang & Wong, 2000). We express equilibrium as a primal problem which is the minimization of a mathematical convex program (Yang, Yagar & Iida, 1994) adapted to the orthotropic case. This program is inspired from the Beckmann objective function (Beckmann, 1952). To solve this program, we use a Lagrangian scheme and a dual method. This technique provides a potential function that explains the flows of traffic over the city, and the over costs generated by the saturated flows. We prove that the solution fits with the User Equilibrium principle (Wardrop, 1952). We apply the model to the case of Paris urban area, where we put an orthotropic network. The commuters are uniformly distributed on the city, and try to reach some destinations located on the ring on-ramps.

Keywords: network flow, continuum modelling, side constraints, traffic assignment, Lagrangian method

1. Introduction

We consider a city in a two-dimensional plane. We note the geographical surface of this city $\mathbb{A}$. The network of the area $\mathbb{A}$ is viewed as a continuum. To illustrate it, one could imagine an American city (like New York, Atlanta, Phoenix etc.) observed from so far away that it would be impossible to distinguish separately the different roads of

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the network. Why do we do this approximation? Because the study of a big network of transportation with graph theory is very difficult to manage on a large urban area. First, the large quantity of data necessary would be very expensive to collect. Second, the computational treatment of these data would be high time consuming, so the traffic scheme could have changed when the outputs would be out. Third, the large quantity of information obtained would be difficult to analyze. Fourth, the sources of error associated to each step of the study would be difficult to control, e.g. errors of data acquisition due to the measuring devices on the network. This kind of study would be expensive (in money and time) and complex. Then, a reasonable simplification of the network consists in consider it as a continuum. This means that instead of representing each road by an arc, we «eliminate» the roads from the mathematical modelling and replace them by a continuous medium. The approximation made in this way can be seen as coarse. Nevertheless, the committed errors are comparable to those the study of a big network with graph theory could lead: hundreds of street represented by the same arc, hundreds of origins and destinations put together in the same node, an origin/destination matrix subject to many data acquisition errors.

2. Road traffic on a large and dense urban area – Continuum network

The surface A can be identified as a subset of the affine space $\mathbb{R}^2$. We consider on A the distance induced by the L^1 norm: between two points A and B which coordinates are (x_A, y_A) and (x_B, y_B), the distance is $d(A,B) = |x_B - x_A| + |y_B - y_A|$. This distance is called orthotropic as far as it is defined from an orthotropic scheme of paths on A. Contrary to an isotropic metric where there is a single shortest path between A and B (the segment $[AB]$), with an orthotropic metric, between two points, several paths can have the same distance (e.g. the green and red paths of the figure 1). We suppose that all the commuters are uniformly distributed in the area A, and try to get to their destination. There are M possible destinations for the customers in the area. We note them $B_m, m \in \{1, \cdots, m\}$. The different traffic flows from the urban area to one destination over all the destinations are called a commodity.

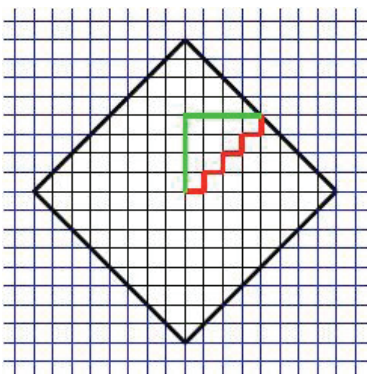


Figure 1. Two paths with same distance in an orthotropic network

3. Demand function of commuters

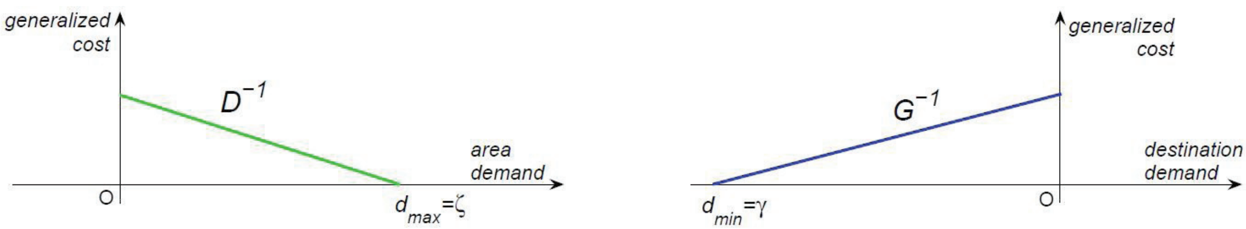
These demand functions have the role in our model of the Origin/ Destination matrix in more “classical” models. The traffic demand from a small area $dxdy$ around the point of the area located by its coordinates (x, y) , to the destination B_m, is the quantity noted $\theta_m(x,y)dxdy$. This quantity is positive over the area $A \backslash B_m$, and strictly negative over B_m. We assume that the demand for each commodity is elastic. This means that for every m, it exists a monotonic decreasing function of the travel cost $\tau_m(x,y)$ from the origin (x,y) to the destination Bm. The travel cost $\tau_m(x,y)$ represents the time cost to reach the destination B_m from the origin (x,y), the transportation cost being neglected.

3.1. Demand function while getting to a destination (generation)

For every m, we note the demand function D_m : $D_m: \forall(x,y) \in A \backslash B_m, D_m(\tau_m(x,y)) = \theta_m(x,y)$. For every m, D_m is supposed to be an affine strictly decreasing function (so $D_m^{-1}$is), the demand at the point (x,y) being set in $[0, \zeta_m(x,y)]$. If ϑ is the demand of travel at a point to the commodity m, the travel cost from this point to the commodity m is $D_m^{-1}(\vartheta)$, and the higher is $D_m^{-1}(\vartheta)$, the lower is ϑ. The economists represent this function in a reverse way from engineers, with the time on the y -axis and the demand on the x -axis. For example, they often use affine functions as on figure 2 (left) to represent the demand.

3.2. Demand function into an area of destination (absorption)

While entering the area of their destination, the commuters feel satisfied. But as in the previous paragraph, we can hypothesize that there will be less demand as the cost of being in the destination's area is big. As the vehicles are absorbed by the destination, we can model this demand function as on figure 2 (right). The demand is negative:



$\theta_m(x,y) \epsilon [\gamma_m(x,y), 0]$.

Figure 2. Affine demand functions

As in an Origin/Destination matrix, there must be conservation equations. This is the aim of the next part.

4. Flow and conservation law

At each point (x,y) of the network , the vector $q_m(x,y) = (q_{m,1}(x,y), q_{m,2}(x,y))$ represents the flow state for the commodity m. Each one of its coordinates is expressed in number of vehicle per length unit (and per hour in the case our static model is built for this period of time). On every disc (or square) Ω included in A, the demand refered to the destination m is equal to the projection of the flow over the normal exterior vectors of Ω, what we can mathematically express by the relation $\iint_\Omega \theta_m(x,y)\,dxdy = \oint_{\partial\Omega} < q_m(x,y), \vec{n} > d\sigma$. If we suppose that the flow q is sufficiently smooth, by using the Green-Riemann theorem, we obtain the relation of conservation in the static case: $\theta_m = div(q_m)$.

5. Road Traffic Physic

5.1. Travel cost

As vehicles in opposite ways do not share the same lane, a crucial difference between traffic flow and fluids is that two opposite traffic movements do not cancel each other out. Figure 3 illustrate this. The flows q_1 and q_2 have nearly the same influence on the travel cost at point P, whereas each one of the flows q_3 and q_4 nearly do not disturb the other. Then, the travel cost at a point where two opposite flows of traffic exist, is the sum of the cost of each flow and not the cost of the sum of the norms of each flow (figure 3).

We consider the two privileged ways of the area. These ways are orthogonal, and give us four possible directions of moving. We explain with the simple example of the figure 4 how we construct the travel cost for the commuters at any point P of the area A. Let the four vectors e_1, e_2, e_3 and e_4 be as in (2.4). The three flows q_1, q_2, and q_3 at point P can be written as $q_1 = q_{1.1}e_1 + q_{1.2}e_2$, $q_2 = q_{2.3}e_3 + q_{2.2}e_2$ and $q_3 = q_{3.3}e_3 + q_{3.4}e_4$ where each of the coordinates $q_{i.j}$ (with $(i,j) \in \{1,2,3,4\}^2$) are positive numbers. As only two orthogonal directions are privileged, it is just as if the travel cost at any point were the sum of the costs of the sum of each flow coordinates on every possible sense of moving (figure 4):

$$C\big(P,(q_1,q_2,q_3)\big) = c_1\big(P,q_{1.1}\big) + c_2\big(P,q_{1.2}+q_{2.2}\big) + c_3\big(P,q_{2.3}+q_{3.3}\big) + c_4\big(P,q_{3.4}\big) \qquad (1)$$

We have written four travel cost functions $c_i, i \in \{1,2,3,4\}$ because we allow the cost to be different for each sense of one determined direction (think about two lanes in one direction and only one lane in the opposite direction, the cost on each direction shouldn't be the same).

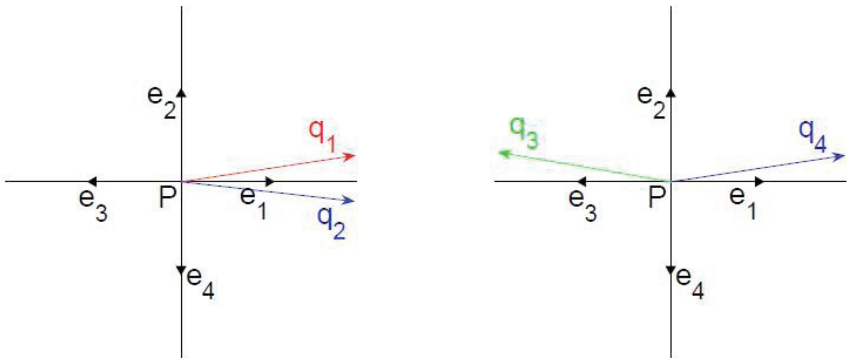


Figure 3. Different flows of traffic at point P

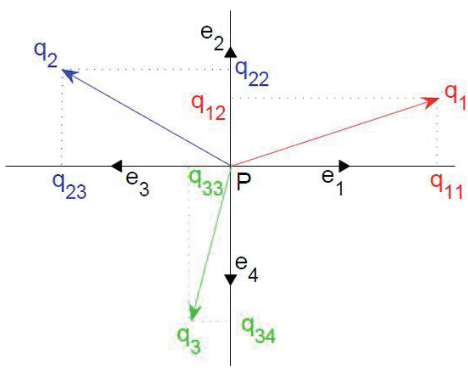


Figure 4. Three different flows at point P

6. Formulation of the equilibrium

For any real number a, we note $R(a)$ or simply a^+, the quantity $max(a, 0)$. The function R is called Ramp function. It is convex (but not strictly convex) and not differentiable at 0.

6.1. Formal approach of the constraint flows and demands set

The functions q_m and θ_m we are looking for must verify conditions that come from the network's topological properties, the capacity constraints and the underlying traffic model. A first constraint is the positivity (which corresponds to a generation of vehicles) and the upper boundedness (which corresponds to a maximum number of generated vehicles per unit area and per hour) of each demand function θ_m at every point of the area which is not a destination point:

$$\forall m \in \{1, \cdots, M\}, \forall (x,y) \in A \backslash B_m, 0 \le \theta_m(x,y) \le \zeta_m(x,y) \tag{2}$$

A second constraint is the negativity (which correspond to absorption of vehicles) and the lower boundedness (which correspond to a maximum number of absorbed vehicles per unit area and per hour) of each demand function θ_m at every point of the area which is a destination point:

$$\forall m \in \{1, \cdots, M\}, \forall (x,y) \in B_m, \gamma_m(x,y) \le \theta_m(x,y) \le 0 \tag{3}$$

A third constraint is the capacity constraint for every direction, the traffic flow cannot exceed it:

$$\forall i \in \{1,2,3,4\}, \forall (x,y) \in A, \textstyle\sum_{m=1}^{M} < q_m(x,y), e_i >^+ \le K_i(x,y) \tag{4}$$

This constraint takes into account the network's topology (an orthotropical network) too.

A fourth constraint is the flow conservation for each destination:

$$\forall m \in \{1, \cdots, M\}, \forall (x,y) \in A, div q_m(x,y) = \theta_m(x,y) \tag{5}$$

A fifth constraint is the border conditions: the traffic flow goes along the boundary of the area and cannot cross it:

$$\forall m \in \{1, \cdots, M\}, \forall (x,y) \in \partial A, < q_m(x,y), \vec{n} > = 0 \tag{6}$$

We note X_0 the set of couples (q, θ) , with $q = (q_m)_{m\in\{1,\cdots,M\}}$ and $\theta = (\theta_m)_{m\in\{1,\cdots,M\}}$, that verify the five constraints listed before.

We won't detail the functional space where the couples (q, θ) are, as far as we will content ourselves with the finite element approximation. We just admit that this space is the Hilbert-Sobolev space $(H_0(div, A))^M \times (L^2(A))^m$, and we do the special remark that the last constraint has been incorporated in the space $(H_0(div, A))^M$, and we observe that X_0 is a bounded closed convex set.

6.2. A Beckmann-like objective function and minimization problem

A commuter of the city wants to reach his destination. The global assignment obeys the first principle of Wardrop (1952), called the "user equilibrium". The problem of user equilibrium in a dense and large urban area with an orthotropic network viewed as a continuum can be formulated with analogy to the Beckmann model for graphs (Beckmann, 1952).We note J_0 the convex function defined by:

$$J_0(q,\theta) = \iint_A \sum_{i=1}^{4} \int_0^{\sum_{m=1}^{M} <q_m(x,y),e_i>^+} c_i(x,y,\xi)d\xi dxdy - \sum_{m=1}^{M}\left[\iint_{A\setminus B_m}\int_0^{\theta_m(x,y)} D_m^{-1}(\vartheta)d\vartheta dxdy + \iint_{B_m}\int_{\theta_m(x,y)}^{0} G_m^{-1}(\vartheta)d\vartheta dxdy\right]. \quad (7)$$

The minimization problem is $(P_0): \min_{(q,\theta)\in X_n} J_0(q,\theta)$. As we already say it, we won't give any mathematical details concerning the minimizer's existence.

6.3. Two important properties of the minimizers

The primitive of a positive function is a strictly convex function. Hence, we obtain that if we have two minimizers $(\bar{q},\bar{\theta})$ and $(\breve{q},\breve{\theta})$ of the functional J_0, then $\bar{\theta}=\breve{\theta}$ (uniqueness of the demand) and $\forall (x,y)\in A, \forall i \in \{1,2,3,4\}, \sum_{m=1}^{M} < \bar{q}_m(x,y), e_i >^+ = \sum_{m=1}^{M} < \breve{q}_m(x,y), e_i >^+$ (uniqueness of the load of traffic in each direction).

7. Approximation of the equilibrium and Lagrangian formulation

The Ramp function is not differentiable. It only has a distributional derivative, called the Heaviside step function, but this type of derivative is too weak for our problem. In the sequel, we will use smooth functions to approximate the Ramp function, so we will be able to derive them. The parameter $\epsilon > 0$ is bound to tend towards 0 (see fig (2.6)). Let us fix $\epsilon > 0$. For every $x \in \mathbb{R}$ we note $r_\epsilon(x) = 0{,}5(x + (x^2+\epsilon^2)^{0{,}5})$ see figure 5. We note $r'_\epsilon = h_\epsilon$ the smooth approximation of the Heaviside function.

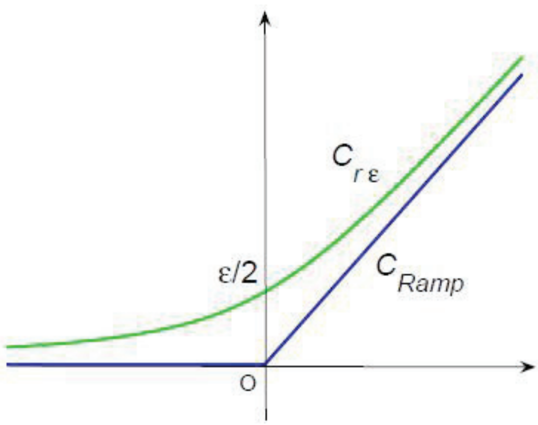


Figure 5. Ramp function approximation

The New problem is:

$$(P_\epsilon): \min_{(q,\theta)\in X_n} J_\epsilon(q,\theta) \quad (8)$$

where X_ϵ is the functional set defined as:

$$\begin{cases} 0 \le \theta_m(x,y) \le \zeta_m(x,y), & \forall m \in \{1,\cdots,M\}, & \forall (x,y) \in A\backslash B_m \\ \gamma_m(x,y) \le \theta_m(x,y) \le 0, & \forall m \in \{1,\cdots,M\}, & \forall (x,y) \in B_m \\ \sum_{m=1}^{M} r_\epsilon(< q_m(x,y), e_i >) \le K_i(x,y) + M\frac{\epsilon}{2}, & \forall i \in \{1,2,3,4\}, & \forall (x,y) \in A \\ divq_m(x,y) = \theta_m(x,y), & \forall m \in \{1,\cdots,M\}, & \forall (x,y) \in A \end{cases} \quad (9)$$

Let us remark that the notations are consistent. We can take $\epsilon = 0$ in the expression of J_ϵ and X_ϵ to have J_0 and X_0. The main fact is that for all $\epsilon > 0$ fixed, the functional J_ϵ is strictly convex and smooth. We note $(q_\epsilon, \theta_\epsilon)$ its unique minimum on X_ϵ.

We now consider the Lagrangian of the problem (P_ϵ):

$L_\epsilon(q,\theta,\phi,\varphi,\chi,\psi,\lambda,\mu) = J_\epsilon(q,\theta) + \langle \phi,\varphi,\chi,\psi,\lambda,\mu | F_\epsilon(q,\theta)\rangle$, where $\langle \ | \ \rangle$ is the scalar product in a product of L^2 spaces:

$$\begin{aligned} &\langle \varphi,\phi,\chi,\psi,\lambda,\mu | F_\epsilon(q,\theta)\rangle \\ &\quad = \sum_{m=1}^{M} \iint_{A\backslash B_m} -\phi_m\theta_m + \varphi_m(\theta_m - \zeta_m) + \iint_{B_m} \chi_m(\gamma_m - \theta_m) + \psi_m\theta_m \\ &\quad + \sum_{i=1}^{4} \iint_A \lambda_i \left[\sum_{m=1}^{M} r_\epsilon(< q_m, e_i >) - K_i - M\frac{\epsilon}{2}\right] + \sum_{m=1}^{M} \iint_A \mu_m(divq_m - \theta_m) \end{aligned} \quad (10)$$

A saddle point $(q_s,\theta_s,\phi_s,\varphi_s,\chi_s,\psi_s,\lambda_s,\mu_s)$ of the Lagrangian is a point such that $\forall(\phi,\varphi,\chi,\psi,\lambda,\mu,\nu), \forall(q,\theta)$, $L_\epsilon(q_s,\theta_s,\phi,\varphi,\chi,\psi,\lambda,\mu) \le L_\epsilon(q_s,\theta_s,\phi_s,\varphi_s,\chi_s,\psi_s,\lambda_s,\mu_s) \le L_\epsilon(q,\theta,\phi_s,\varphi_s,\chi_s,\psi_s,\lambda_s,\mu_s)$.

In particular, it is possible to prove that if $(q_s,\theta_s,\phi_s,\varphi_s,\chi_s,\psi_s,\lambda_s,\mu_s)$ is a saddle point of the Lagrangian, then automatically (q_s,θ_s) is a minimum of J_ϵ on X_ϵ (then $(q_s,\theta_s) = (q_\epsilon,\theta_\epsilon)$). Now, if we note a saddle point of the Lagrangian $(q_\epsilon,\theta_\epsilon,\phi_\epsilon,\varphi_\epsilon,\chi_\epsilon,\psi_\epsilon,\lambda_\epsilon,\mu_\epsilon)$, the relation: $\forall(\hat{q},\hat{\theta}), D_{(q,\theta)} L_\epsilon(q_\epsilon,\theta_\epsilon,\phi_\epsilon,\varphi_\epsilon,\chi_\epsilon,\psi_\epsilon,\lambda_\epsilon,\mu_\epsilon \mid \hat{q},\hat{\theta}) = 0$ gives us that: $\forall m$,

$$\begin{aligned} &\sum_{i=1}^{4} \left[\lambda_{\epsilon,i} + c_i(x,y,\sum_{m=1}^{M} r_\epsilon(< q_{\epsilon,m}, e_i >)\right] h_\epsilon(< q_{\epsilon,m}, e_i >) e_i = \nabla\mu_{\epsilon,m} \\ &-D_m^{-1}(\theta_{\epsilon,m}) - \phi_{\epsilon,m} + \varphi_{\epsilon,m} - \mu_{\epsilon,m} = 0 \text{ on } A\backslash B_m \\ &G_m^{-1}(\theta_{\epsilon,m}) - \chi_{\epsilon,m} + \psi_{\epsilon,m} - \mu_{\epsilon,m} = 0 \text{ on } B_m \end{aligned} \quad (11)$$

8. Cost of used/unused paths

If we consider that the functions λ are associated to an over cost, the cost of a used path *p* between a couple Origin/Destination (O, D_m) is:

$$\begin{aligned} Cost(p,m) = &\int_p \left[\lambda_{\epsilon,1} + c_1(x,y,\sum_{m=1}^{M} r_\epsilon(< q_{\epsilon,m}, e_1 >)\right] h_\epsilon(< q_{\epsilon,m}, e_1 >)\,|dx| \\ &+ \int_p \left[\lambda_{\epsilon,2} + c_2(x,y,\sum_{m=1}^{M} r_\epsilon(< q_{\epsilon,m}, e_2 >)\right] h_\epsilon(< q_{\epsilon,m}, e_2 >)\,|dy| \\ &+ \int_p \left[\lambda_{\epsilon,3} + c_3(x,y,\sum_{m=1}^{M} r_\epsilon(< q_{\epsilon,m}, e_3 >)\right] h_\epsilon(< q_{\epsilon,m}, e_3 >)\,|dx| \\ &+ \int_p \left[\lambda_{\epsilon,4} + c_4(x,y,\sum_{m=1}^{M} r_\epsilon(< q_{\epsilon,m}, e_4 >)\right] h_\epsilon(< q_{\epsilon,m}, e_4 >)\,|dy| \end{aligned} \tag{12}$$

Hence, by using the property that for any used path, $|dx| = \text{sgn}(< q_{\epsilon,m}, e_1 >)dx$, and $|dy| = \text{sgn}(< q_{\epsilon,m}, e_2 >)dy$, we can write:

$$Cost(p,m) = \int_p \langle \nabla\mu_{\epsilon,m} \,|\, \begin{matrix} dx \\ dy \end{matrix} \rangle = \mu_{\epsilon,m}(D_m) - \mu_{\epsilon,m}(O) \tag{13}$$

which does not depend on the used path p.

Now, if we consider an unused path p between the couple (O, D_m) , we have the relations $|dx| \geq \text{sgn}(< q_{\epsilon,m}, e_1 >)dx$ and $|dy| \geq \text{sgn}(< q_{\epsilon,m}, e_2 >)dy$, which gives us that $Cost(p,m) \geq \mu_{\epsilon,m}(D_m) - \mu_{\epsilon,m}(O)$.

9. Results analysis Beckmann, 1952

We have implemented an Uzawa algorithm. Each step of it need to minimize a function, we have done it with a Newton-Raphson Algorithm. We chose the city of Paris. The aim was to apply our model to a surface which is neither an ellipse nor a rectangle, which are too smooth. Moreover, the Seine offers a natural obstacle that can helps to understand the behavior of our algorithm. We put only eight destinations (there are sixteen in reality), but it doesn't change the traffic physic described in our model (figure 6). The figure 8 represents the solutions we obtain for the demand functions (θ_m) , the figure 9 represents the load of traffic for the direction e_1 : $\sum_{m=1}^{M} < q_m, e_1 >^+$.

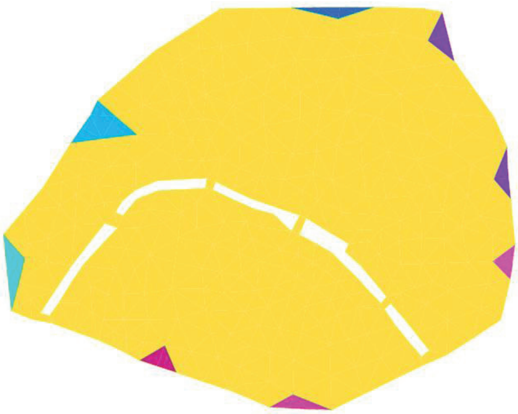


Figure 6. Paris seen as a continuum

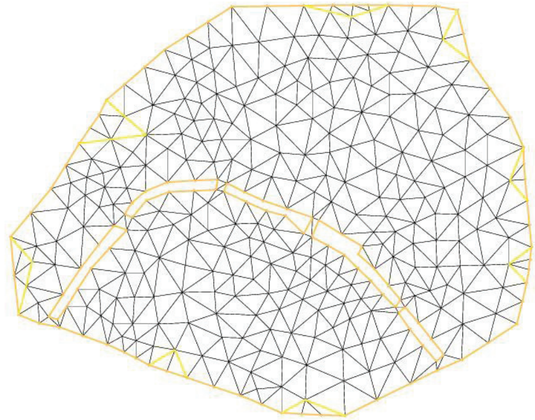


Figure 7. Meshing

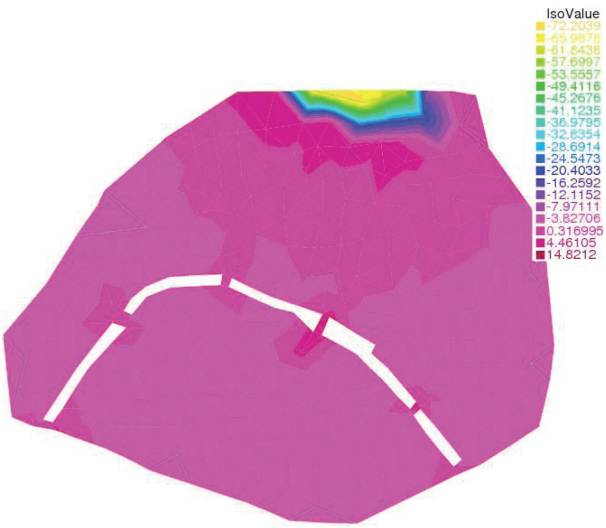


Figure 8. Generation function for the North destination

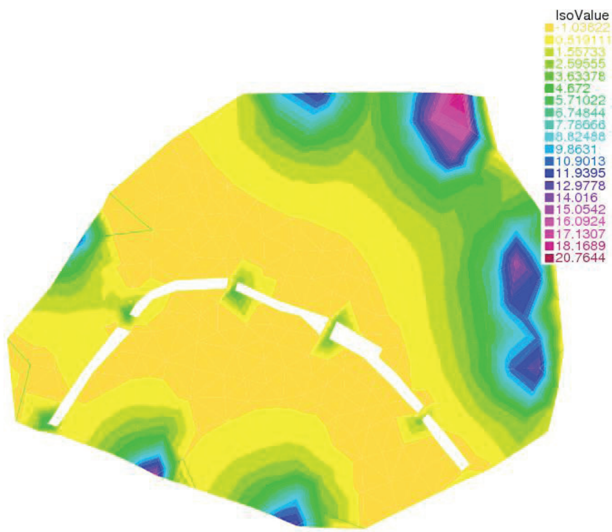


Figure 9. traffic load in the direction West to East

10. Conclusion and further investigations

In this study, a traffic model in dense urban area has been developed. It is based on the following few elements: the network is orthotropical, it is approximated by a continuum, the network's offer is defined with the help of side constraints, the commuters of the area try to get to one of the possible destinations. A mathematical program (primal formulation) for the model has been developed and proved to satisfy the user equilibrium. The dual coefficients associated to the over-cost and the potential cost in the area have also been discussed. An Uzawa algorithm has been implemented to solve the problem in a numerical example.

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